REMOVABLE SINGULARITIES AND BUBBLING OF HARMONIC MAPS AND BIHARMONIC MAPS

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ABSTRACT. In this paper, by using Moser's iteration tequnique, we show some removable singularity theorem of the tension field for biharmonic maps into manifolds of non-positive curvature, and the bubbling theorem for biharmonic maps and also harmonic maps.

1. INTRODUCTION

Harmonic maps play a central roll in variational problems, which are, by definition, critical maps of the energy functional $E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2\, v_g$ for smooth maps φ of (M, g) into (N, h). By extending the notion of harmonic maps, in 1983, J. Eells and L. Lemaire [7] proposed the problem to consider the polyharmonic, i.e., k-harmonic maps which are critical maps of the functional

$$E_k(\varphi) = \frac{1}{2}\int_M |(d+\delta)^k \varphi|^2\, v_g, \quad (k = 1, 2, \cdots).$$

After G.Y. Jiang [19] studied the first and second variation formulas of E_2 for $k = 2$, whose critical maps are called biharmonic maps, there have been extensive studies in this area (for instance, see [3], [23], [24], [26], [25], [15], [16]. [18], [17], [29], etc.). Harmonic maps are always biharmonic maps by definition.

The theory of regularity for harmonic maps and biharmonic ones has a long history. We summarize it briefly:

In 1981, Sack and Uhlenbeck showed ([28], see also [20]) that

If $\varphi : \mathbf{B}^2\backslash\{o\} \to (N, h)$ is harmonic with finite energy, then φ extends to a smooth harmonic map $\varphi : D \to (N, h)$, where $\mathbf{B}^2$ is a 2-dimensional unit disc with the origin o, and (N, h) is an arbitrary Riemannian manifold.

2000 *Mathematics Subject Classification.* primary 58E20, secondary 53C43.

Key words and phrases. harmonic map, biharmonic map, removable singularity, bubbling.

Supported by the Grant-in-Aid for the Scientific Reserch, (A), No. 19204004, (C) No. 21540207, Japan Society for the Promotion of Science.

In 1982, Schoen and Uhlenbeck showed ([31], see also [30]) that

(*a*) *for any energy minimizing map* $\varphi \in L_1^2(M,N)$, *the Hausdorff dimension of the singular set* $\mathcal{S}$ *of* φ *is smaller than or equal to* $\dim M - 3$, *and* $\mathcal{S}$ *is discrete if* $\dim M = 3$, *and* φ *is a smooth harmonic map if* $\dim M = 2$.

(*b*) *Furthermore, if the curvature of* (N,h) *is non-positive, any energy minimizing map in* $L_1^2(M,N)$ *is a smooth harmonic map.*

In 1984, Eells and Polking [10] showed that,

let $\varphi \in L^2_{1,loc}(M,N)$ *be weakly harmonic on the complement of a polar set in* M. *Then,* φ *is weakly harmonic on* M,

where notice ([9], p. 397) that

φ *is harmonic if it is weakly harmonic and continuous.*

On the contrary, in 1995, Rivière [27] gave examples of

weakly harmonic maps in $L_1^2(\mathbf{B}^3, S^2)$ *which are discontinuous everywhere in* $\mathbf{B}^3$.

See the regularity works due to Hildebrant, Kaul and Widman [14], Bethuel [2], Evans [11], Helein [13], and also Struwe [32].

For the regularity theory of biharmonic maps, Chang, Wang and Yang [5] showed that

(1) *any biharmonic map of a four dimensional disc into the standard unit sphere* (S^n, g_0) *is Hölder continuous,*

(2) *a stationally biharmonic map of* $\mathbf{B}^m$ $(m \geq 5)$ *into* (S^n, g_0) *is Hölder continuous except on a set of* $(m-4)$*-dimensional Hausdorff measure zero, and*

(3) *a weak biharmonic map which is continuous is smooth.*

Struwe [33] showed that

any stationary biharmonic map satisfying some growth condition of $\mathbf{B}^m$ *into any Riemannian manifold is Hl̈der continuous, in particular, smooth out of a set of* $(m-4)$*-dimensional Hausdorff measure zero.*

In this paper, we will show the following.

Theorem 1.1. (*cf.* **Theorem 4.2**) *Assume that* (M,g) *is a compact Riemannian manifold, and the sectional curvature of* (N,h) *is non-positive, and there exists a finite set* $\mathcal{S}$ *of points in* M *such that* φ :

$(M\backslash\mathcal{S}, g) \to (N, h)$ is a biharmonic map and have the finite bienergy:

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2\, v_g < \infty. \tag{1.1}$$

Then, $|\tau(\varphi)|$ can be extended continuously to (M, g).

The theory of bubbling phenomena of harmonic maps has begun at Sacks and Uhlenbeck [31] and extended to several variational problems including Yang-Mills theory (see Freed and Uhlenbeck [12])). For the bubbling phenomena of biharmonic maps, we will show

Theorem 1.2. (*cf.* **Theorem 5.1**) *Let (M, g) and (N, h) be two compact Riemannian manifolds. For every positive constant $C > 0$, let us consider a family of biharmonic maps of (M, g) into (N, h),*

$$\begin{aligned}\mathcal{F} = \Big\{\varphi :&(M, g) \to (N, h),\ \textit{biharmonic}\ | \\ &\int_M |\, d\varphi\,|^m\, v_g \leq C \ \textit{and}\ \int_M |\tau(\varphi)|^2\, v_g \leq C\Big\},\end{aligned} \tag{1.2}$$

where $m = \dim M$. Then, any sequence in $\mathcal{F}$ causes a **bubbling***: Namely, for any sequence $\{\varphi_i\} \in \mathcal{F}$, there exist a finite set $\mathcal{S}$ in M, say, $\mathcal{S} = \{x_1, \cdots, x_\ell\}$, and a smooth biharmonic map $\varphi_\infty : (M\backslash\mathcal{S}, g) \to (N, h)$ suth that,*

(1) *a subsequence φ_{i_j} converges φ_∞ in the C^∞-topology on $M\backslash\mathcal{S}$, as $j \to \infty$, and*

(2) *the Radon measures $|d\varphi_{i_j}|^m\, v_g$ converges to a measure*

$$|d\varphi_\infty|^m\, v_g + \sum_{i=1}^{\ell} a_k\, \delta_{x_k}, \tag{1.3}$$

as $j \to \infty$. Here a_k is a constant, and δ_{x_k} is the Dirac measure whose support is $\{x_k\}$ $(k = 1 \cdots, \ell)$.

As an application, we have the bubbling theorem for harmonic maps (cf. Theorem 5.2).

Acknowledgement. We would like to express our gratitude to Professor H. Naito for his useful suggestions and comments during the preparations of this paper.

2. Preliminaries

In this section, we prepare materials for the first and second variation formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map $\varphi : (M,g) \to (N,h)$, of a compact Riemannian manifold (M,g) into another Riemannian manifold (N,h), which is an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi)\, v_g,$$

where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of φ. That is, for any variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\frac{d}{dt}\bigg|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0, \tag{2.1}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ which is given by $V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$, $(x \in M)$, and the *tension field* is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined frame field on (M,g), and $B(\varphi)$ is the second fundamental form of φ defined by

$$\begin{aligned} B(\varphi)(X,Y) &= (\widetilde{\nabla} d\varphi)(X,Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y) \\ &= \nabla^N_{d\varphi(X)} d\varphi(Y) - d\varphi(\nabla_X Y), \end{aligned} \tag{2.2}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Furthermore, ∇, and ∇^N, are connections on TM, TN of (M,g), (N,h), respectively, and $\overline{\nabla}$, and $\widetilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2.1), φ is harmonic if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that φ is harmonic. Then,

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(\varphi_t) = \int_M h(J(V), V) v_g, \tag{2.3}$$

where J is an ellptic differential operator, called *Jacobi operator* acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta} V - \mathcal{R}(V), \tag{2.4}$$

where $\overline{\Delta} V = \overline{\nabla}^* \overline{\nabla} V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} V - \overline{\nabla}_{\nabla_{e_i} e_i} V\}$ is the *rough Laplacian* and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}V = \sum_{i=1}^m R^N(V, d\varphi(e_i)) d\varphi(e_i)$, and R^N is the curvature tensor of (N,h) given by $R^N(U,V) = \nabla^N{}_U \nabla^N{}_V - \nabla^N{}_V \nabla^N{}_U - \nabla^N{}_{[U,V]}$ for U, $V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire [7] proposed polyharmonic (k-harmonic) maps and Jiang [19] studied the first and second variation formulas of biharmonic maps. Let us consider the *bienergy functional* defined by

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g, \tag{2.5}$$

where $|V|^2 = h(V,V)$, $V \in \Gamma(\varphi^{-1}TN)$.

Then, the first variation formula of the bienergy functional is given as follows.

Theorem 2.1. (*the first variation formula*)

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V) v_g. \tag{2.6}$$

Here,

$$\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi)), \tag{2.7}$$

which is called the bitension field *of* φ*, and* J *is given in* (2.4).

Definition 2.2. *A smooth map* φ *of* M *into* N *is said to be* biharmonic *if* $\tau_2(\varphi) = 0$.

3. The Bochner-type estimation for the tension field of a biharmonic map

In this section, we give the Bochner-type estimations for the tension fields of biharmonic maps into a Riemannian manifold (N,h) of non-positive curvature.

Lemma 3.1. *Assume that the sectional curvature of* (N,h) *is non-positive, and* $\varphi : (M\backslash\mathcal{S}, g) \to (N,h)$ *is a biharmonic mapping for some closed set* $\mathcal{S}$ *of* M*. Then, it holds that*

$$\Delta\,|\tau(\varphi)|^2 \geq 2\,|\overline{\nabla}\tau(\varphi)|^2 \tag{3.1}$$

at each point in $M\backslash\mathcal{S}$*. Here,* Δ *is the Laplace-Beltrami operator, i.e., the negative Laplacian of* (M,g) *acting on* $C^\infty(M)$.

Proof. Let us take a locally defined orthonormal frame field $\{e_i\}_{i=1}^m$ on $M\backslash\mathcal{S}$, and $\varphi : (M\backslash\mathcal{S}, g) \to (N,h)$, a biharmonic map. Then, for

$V := \tau(\varphi) \in \Gamma(\varphi^{-1}TN)$, we have

$$\begin{aligned}
\frac{1}{2}\,\Delta\,|V|^2 &= \frac{1}{2}\sum_{i=1}^{m}\Big\{{e_i}^2\,|V|^2 - \nabla_{e_i}e_i\,|V|^2\Big\} \\
&= \sum_{i=1}^{m}\Big\{e_i\,h(\overline{\nabla}_{e_i}V,V) - h(\overline{\nabla}_{\nabla_{e_i}e_i}V,V)\Big\} \\
&= \sum_{i=1}^{m}\Big\{h(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\,V,V) - h(\overline{\nabla}_{\nabla_{e_i}e_i}\,V,V)\Big\} \\
&\qquad + \sum_{i=1}^{m} h(\overline{\nabla}_{e_i}V,\overline{\nabla}_{e_i}V) \\
&= h(-\overline{\Delta}\,V,V) + |\overline{\nabla}V|^2 \\
&= h(-\mathcal{R}(V),V) + |\overline{\nabla}V|^2 \\
&\geq |\overline{\nabla}V|^2, \qquad (3.2)
\end{aligned}$$

because for the second last equality, we used $\overline{\Delta}V - \mathcal{R}(V) = J(V) = 0$ for $V = \tau(\varphi)$, due to the biharmonicity of $\varphi:\,(M\backslash\mathcal{S},g)\to(N,h)$, and for the last inequality of (3.2), we used

$$h(\mathcal{R}(V),V) = \sum_{i=1}^{m} h(R^N(V,\varphi_*e_i)\varphi_*e_i,V) \leq 0 \qquad (3.3)$$

since the sectional curvature of (N,h) is non-positive. $\square$

By Lemma 3.1, we have

Lemma 3.2. *Under the same assumptions as Lemma 3.1, we have*

$$|\tau(\varphi)|\;\Delta\,|\tau(\varphi)| \geq 0. \qquad (3.4)$$

Proof. Due to Lemm 3.1, we have

$$\begin{aligned}
2\,|\overline{\nabla}\tau(\varphi)|^2 &\leq \Delta\,|\tau(\varphi)|^2 \\
&= 2\,|\tau(\varphi)|\;\Delta\,|\tau(\varphi)| + 2\,|\,\nabla\,|\tau(\varphi)|\,|^2. \qquad (3.5)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|\tau(\varphi)|\,\Delta\,|(\tau(\varphi)| &\geq |\overline{\nabla}\tau(\varphi)|^2 - |\,\nabla\,|\tau(\varphi)|\,|^2 \\
&\geq 0. \qquad (3.6)
\end{aligned}$$

Here, to see the last inequality of (3.6), it suffices to notice that for all $V \in \Gamma(\varphi^{-1}TN)$,

$$|\overline{\nabla}V| \geq |\;\nabla\,|V|\;| \qquad (3.7)$$

which follows from that

$$
\begin{aligned}
|V| \, | \nabla |V| \, | &= \frac{1}{2} \, | \nabla |V|^2 \, | \\
&= \frac{1}{2} \, |\nabla \, h(V,V) \, | \\
&= | \, h(\overline{\nabla} V, V) \, | \\
&\leq | \, \overline{\nabla} V \, | \, |V|. \qquad (3.8)
\end{aligned}
$$

We have Lemma 3.2. □

Then, by using Moser's iteration technique due to this Lemma 3.2, we have the following theorem.

Theorem 3.3. *Assume that (M,g) is a compact Riemannian manifold, and the sectional curvature of (N,h) is non-positive. Then, there exists a positive constant $C > 0$ depending only on $\dim M$ such that for every biharmonic mapping $\varphi : (M\backslash\mathcal{S}, g) \to (N,h)$ with $\mathcal{S} = \{x_1, \cdots, x_\ell\}$, every positive number $r > 0$ and each point $x_i \in \mathcal{S}$,*

$$
\sup_{B_r(x_i)} |\tau(\varphi)| \leq \frac{C}{r^{m/2}} \int_{B_{2r}(x_i)} |\tau(\varphi)|^2 v_g, \qquad (3.9)
$$

where $B_r(x_i) = \{x \in M; \, r(x,x_i) < r\}$ is the metric ball in (M,g) around x_i of radius r, for every sufficient small $r > 0$ in such a way that $B_r(x_i) \cap B_r(x_j) = \emptyset$ $(i \neq j)$.

4. Moser's iteration technique and proof of Theorem 3.3

(*The first step*) For a fixed point $x_i \in \mathcal{S}$, and for every $0 < \rho_1 < \rho_2 < \infty$, we first take a cutoff C^∞ function η on M (for instance, see [21]) satisfying that

$$
\begin{cases}
0 \leq \eta(x) \leq 1 & (x \in M), \\
\quad 1 & (x \in B_{\rho_1}(x_i)), \\
\quad 0 & (x \notin B_{\rho_2}(x_i)), \\
|\nabla \eta| \leq \dfrac{2}{\rho_2 - \rho_1} & (x \in M).
\end{cases} \qquad (4.1)
$$

For $2 \leq p < \infty$, multiply $|\tau(\varphi)|^{p-2}\,\eta^2$ to both hand sides of the inequality (3.4) in Lemma 3.2, and integrate over M, we have

$$\begin{aligned}
0 &\leq \int_M |\tau(\varphi)|^{p-1}\,\eta^2\,\Delta\left(|\tau(\varphi)|\right) v_g \\
&= -\int_M g(\nabla(|\tau(\varphi)|^{p-1}\,\eta^2), \nabla\,|\tau(\varphi)|)\, v_g \\
&= -(p-1)\int_M |\tau(\varphi)|^{p-2}\,\eta^2\,|\,\nabla(|\tau(\varphi)|)\,|^2\, v_g \\
&\quad - 2\int_M |\tau(\varphi)|^{p-1}\,\eta\, g(\nabla(|\tau(\varphi)|), \nabla\eta)\, v_g \\
&= -\frac{4(p-1)}{p^2}\int_M |\nabla(|\tau(\varphi)|^{p/2})|^2\,\eta^2\, v_g \\
&\quad - \frac{4}{p}\int_M g(\eta\,\nabla(|\tau(\varphi)|^{p/2}), |\tau(\varphi)|^{p/2}\,\nabla\eta)\, v_g. \qquad (4.2)
\end{aligned}$$

Therefore, by using Young's inequality, we have, for every positive real number $\epsilon > 0$,

$$\begin{aligned}
\int_M |\,\nabla(\,|\tau(\varphi)\,|^{p/2}\,)|^2\,\eta^2\, v_g &\leq \frac{p}{p-1}\int_M g(\eta\,\nabla(|\tau(\varphi)|^{p/2}), |\tau(\varphi)|^{p/2}\,\nabla\eta)\, v_g \\
&\leq \frac{p}{2(p-1)}\left\{\epsilon\int_M \eta^2\,|\,\nabla(|\tau(\varphi)|^{p/2})|^2\, v_g \right. \\
&\qquad \left. + \frac{1}{\epsilon}\int_M |\tau(\varphi)|^p\,|\nabla\eta|^2\, v_g\right\}. \qquad (4.3)
\end{aligned}$$

By (4.3), we have

$$\begin{aligned}
&\left(1 - \frac{p}{2(p-1)}\epsilon\right)\int_M \eta^2\,|\,\nabla(|\tau(\varphi)|^{p/2})|^2\, v_g \\
&\qquad \leq \frac{p}{2(p-1)}\frac{1}{\epsilon}\int_M |\tau(\varphi)|^p\,|\nabla\eta|^2\, v_g. \qquad (4.4)
\end{aligned}$$

By choosing $\epsilon = \frac{p-1}{p}$ in (4.4), we have

$$\int_M \eta^2\,|\,\nabla(|\tau(\varphi)|^{p/2})|^2\, v_g \leq \frac{p^2}{(p-1)^2}\int_M |\tau(\varphi)|^p\,|\nabla\eta|^2\, v_g. \qquad (4.5)$$

Here, by using

$$\nabla(|\tau(\varphi)|^{p/2}\,\eta) = \eta\,\nabla(|\tau(\varphi)|^{p/2}) + |\tau(\varphi)|^{p/2}\,\nabla\eta,$$

$|A+B|^2 \leq 2\,|A|^2 + 2\,|B|^2$ and (4.5), and then, by (4.1), we have

$$\begin{aligned}\int_M |\,\nabla(|\tau(\varphi)|^{p/2}\,\eta)\,|^2\,v_g &\leq 2\int_M \eta^2\,|\,\nabla(|\tau(\varphi)|^{p/2})\,|^2\,v_g \\ &\quad + 2\int_M |\tau(\varphi)|^p\,|\nabla\eta|^2\,v_g \\ &\leq 4\,\frac{p^2}{(p-1)^2}\int_M |\tau(\varphi)|^p\,|\nabla\eta|^2\,v_g \\ &\leq \frac{p^2}{(p-1)^2}\frac{16}{(\rho_2-\rho_1)^2}\int_{B_{\rho_2}(x_i)} |\tau(\varphi)|^p\,v_g.\end{aligned} \tag{4.6}$$

For the left hand side of (4.6), let us recall the Sobolev embedding theorem (cf. [1], p. 55; [12], p. 95):

$$H^2_1(M) \subset L^\gamma(M), \tag{4.7}$$

where $\gamma := \frac{m}{m-2}$, i.e., there exists a pocitive constant $C > 0$ such that

$$\left(\int_M |\,f\,|^\gamma\,v_g\right)^{1/\gamma} \leq C\left(\int_M |\,\nabla f\,|^2\,v_g\right)^{1/2} \qquad (\forall\, f \in H^2_1(M)). \tag{4.8}$$

In the case $m = \dim M = 2$, (4.7) and (4.8) still hold, but the left hand side of (4.8) should be replaced into the supremum norm, $\sup_M |f|$.

Therefore, we have

$$\begin{aligned}\int_M |\,\nabla(|\tau(\varphi)|^{p/2}\,\eta)\,|^2\,v_g &\geq \frac{1}{C}\left(\int_M \left\{\,|\tau(\varphi)|^{p/2}\,\eta\right\}^\gamma\,v_g\right)^{2/\gamma} \\ &\geq \frac{1}{C}\left(\int_{B_{\rho_1}(x_i)} \left(|\tau(\varphi)|^{p/2}\right)^\gamma\,v_g\right)^{2/\gamma},\end{aligned} \tag{4.9}$$

where we used (4.1).

Thus, together with (4.6) and (4.9), we have

Lemma 4.1. *Assume that (M,g) is a compact Riemannian manifold, the sectional curvature of (N,h) is non-positive, and $\varphi : (M\backslash\mathcal{S}, g) \to (N,h)$ is a biharmonic mapping, where $\mathcal{S} = \{x_1, \cdots, x_\ell\} \subset M$. Then, for each $0 < \rho_1 < \rho_2 < \infty$, and $2 \leq p < \infty$, it holds that for each $i = 1, \cdots, \ell$,*

$$\begin{aligned}\left(\int_{B_{\rho_1}(x_i)} \left(|\tau(\varphi)|^{p/2}\right)^\gamma\,v_g\right)^{1/\gamma} &\leq \frac{p}{p-1}\,\frac{C'}{\rho_2-\rho_1}\,\times \\ &\qquad \times\left(\int_{B_{\rho_2}(x_i)} \left(|\tau(\varphi)|^{p/2}\right)^2\,v_g\right)^{1/2},\end{aligned} \tag{4.10}$$

where $C' = 4\sqrt{C}$, *and* $C > 0$ *is the Sobolev constant in* (4.8) *and* $\gamma := \frac{2m}{m-2}$, $m = \dim M$. *In the case* $m = \dim M = 2$, *the left hand side of* (4.10) *is replaced into* $\sup_{B_{\rho_1}(x_i)} |\tau(\varphi)|^{p/2}$.

(*The second step*) Here, let us define

$$\begin{cases} \overline{\gamma} := \dfrac{m}{m-2} = \dfrac{1}{2}\gamma, \\ p_k := 2\,\overline{\gamma}^{k-1} \ \to \infty \quad (k \to \infty), \\ r_k := \left(1 + \dfrac{1}{2^{k-1}}\right) r \ \to r \quad (k \to \infty), \end{cases} \tag{4.11}$$

and in (4.10), let us put

$$\begin{cases} p := p_k, \\ \rho_1 := r_{k+1}, \\ \rho_2 := r_k. \end{cases}$$

Then, we have

$$\begin{cases} \dfrac{p\,\gamma}{2} = p_k\,\overline{\gamma} = 2\,\overline{\gamma}^k = p_{k+1}, \\ \rho_2 - \rho_1 = r_k - r_{k+1} = \left(\dfrac{1}{2^{k-1}} - \dfrac{1}{2^k}\right) r = \dfrac{1}{2^k}\, r, \end{cases} \tag{4.12}$$

so that (4.10) can be rewritten as follows.

$$\begin{aligned} \left(\int_{B_{r_{k+1}}(x_i)} |\tau(\varphi)|^{p_{k+1}}\, v_g \right)^{1/\gamma} \leq \frac{2\,\overline{\gamma}^{k-1}}{2\,\overline{\gamma}^{k-1} - 1}\, \frac{2^k}{r} \,\times& \\ \times \left(\int_{B_{r_k}(x_i)} |\tau(\varphi)|^{p_k}\, v_g \right)^{1/2}&. \end{aligned} \tag{4.13}$$

By taking $\frac{1}{\overline{\gamma}^{k-1}}$ power of (4.13), we have

$$\begin{aligned} \|\tau(\varphi)\|_{L^{p_{k+1}}(B_{r_{k+1}}(x_i))} \leq \left(\frac{2\,\overline{\gamma}^{k-1}}{2\,\overline{\gamma}^{k-1} - 1} \right)^{2/p_k} \frac{2^{(k/\overline{\gamma}^{k-1})}}{r^{(1/\overline{\gamma}^{k-1})}} \,\times& \\ \times\, \|\tau(\varphi)\|_{L^{p_k}(B_{r_k}(x_i))}& \end{aligned} \tag{4.14}$$

since, for the power of the left hand side of (4.13), we calculated as

$$\frac{1}{\gamma}\, \frac{1}{\overline{\gamma}^{k-1}} = \frac{1}{2\,\overline{\gamma}\ \overline{\gamma}^{k-1}} = \frac{1}{2\,\overline{\gamma}^k} = \frac{1}{p_{k+1}}.$$

(*The third step*) Now iterate (4.14), then we have

$$\|\tau(\varphi)\|_{L^{p_{k+1}}(B_{r_{k+1}}(x_i))} \leq \prod_{k=1}^{\infty} \left(\frac{2\,\overline{\gamma}^{k-1}}{2\,\overline{\gamma}^{k-1}-1}\right)^{2/p_k} \frac{2^{(k/\overline{\gamma}^{k-1})}}{r^{(1/\overline{\gamma}^{k-1})}} \times \\ \times \|\tau(\varphi)\|_{L^2(B_{2r}(x_i))} \tag{4.15}$$

since $p_1 = 2$ and $r_1 = 2r$. Here, we notice that

$$\prod_{k=1}^{\infty} \frac{1}{r^{(1/\overline{\gamma}^{k-1})}} = \frac{1}{r^{(\sum_{k=1}^{\infty} 1/\overline{\gamma}^{k-1})}} = \frac{1}{r^{m/2}} \tag{4.16}$$

since

$$\sum_{k=1}^{\infty} \frac{1}{\overline{\gamma}^{k-1}} = \frac{1}{1-\frac{1}{\overline{\gamma}}} = \frac{1}{1-\frac{m-2}{m}} = \frac{m}{2}.$$

Notice also that

$$\prod_{k=1}^{\infty} \frac{1}{(2\,\overline{\gamma}^{k-1}-1)^{2/p_k}} \leq 1 \tag{4.17}$$

since $2\,\overline{\gamma}^{k-1} - 1 > 2 - 1 = 1$ when $\overline{\gamma} = m/(m-2) > 1$ $(m \geq 3)$, and the left hand side of (4.17) is equal to 1 when $\overline{\gamma} = \infty$ $(m = 2)$. And also notice that

$$\prod_{k=1}^{\infty} 2^{(k/\overline{\gamma}^{k-1})} = 2^{\sum_{k=1}^{\infty} \frac{k}{\overline{\gamma}^{k-1}}} < \infty, \tag{4.18}$$

$$\prod_{k=1}^{\infty} (2\,\overline{\gamma})^{2(k-1)/p_k} = \gamma^{2\sum_{k=1}^{\infty} \frac{k-1}{p_k}} = \gamma^{\sum_{k=1}^{\infty} \frac{k-1}{\overline{\gamma}^{k-1}}} < \infty. \tag{4.19}$$

Therefore, (4.15) turns out that

$$\|\tau(\varphi)\|_{L^{p_{k+1}}(B_{r_{k+1}}(x_i))} \leq C''\,\frac{1}{r^{m/2}}\,\|\tau(\varphi)\|_{L^2(B_{2r}(x_i))} \tag{4.20}$$

for some positive constant C'' depending only on $m = \dim M$.

(*The fourth step*) Now, let k tend to infinity. Then, by (4.11), the norm $\|\tau(\varphi)\|_{L^{p_{k+1}}(B_{r_{k+1}}(x_i))}$ tends to

$$\|\tau(\varphi)\|_{L^\infty(B_r(x_i))} = \sup_{B_r(x_i)} |\tau(\varphi)|.$$

Thus, we obtain

$$\sup_{B_r(x_i)} |\tau(\varphi)| \leq \frac{C''}{r^{m/2}}\,\|\tau(\varphi)\|_{L^2(B_{2r}(x_i))}, \tag{4.21}$$

which is the desired inequality (3.9). We have Theorem 3.3. $\square$

Due to Theorem 3.3 , we have immediately

Theorem 4.2. *Assume that (M, g) is a compact Riemannian manifold, and the sectional curvature of (N, h) is non-positive, and there exists a finite set $\mathcal{S}$ of points in M, say $\mathcal{S} = \{x_1, \cdots, x_\ell\}$, such that $\varphi : (M\backslash\mathcal{S}, g) \to (N, h)$ is a biharmonic map and have the finite bienergy:*

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2\, v_g < \infty. \tag{4.22}$$

Then, the norm $|\varphi|$ of the tension field $\tau(\varphi)$ is bounded on M. So, $|\tau(\varphi)|$ has a unique continuous extension on (M, g).

Remark 4.3. *If we assume the boundednes of $|d\varphi|$ on M added to the assumptions of Theorem 4.2, then φ can be uniquely extended to a biharmonic map of (M, g) into (N, h). However, notice here that the function $\varphi(z) := \frac{1}{z}$ on $(\mathbb{C} \cup \{\infty\})\backslash\{0\}$ cannot be extended to $\mathbb{C} \cup \{\infty\}$. Indeed, it is holomorphic and harmonic, but $|d\varphi|$ is not bounded on $(\mathbb{C} \cup \{\infty\})\backslash\{0\}$.*

5. Bubbling Theorem of Biharmonic Maps

We have the following bubbling theorem for biharmonic maps.

Theorem 5.1. (*Bubbling for Biharmonic Maps*) *Let (M, g) and (N, h) be two compact Riemannian manifolds. For every positive constant $C > 0$, consider a family of biharmonic maps of (M, g) into (N, h),*

$$\mathcal{F} = \Big\{\varphi :(M, g) \to (N, h),\ \textit{biharmonic}\ | \\ \int_M |\, d\varphi\,|^m\, v_g \leq C \ \textit{and}\ \int_M |\tau(\varphi)|^2\, v_g \leq C\Big\}, \tag{5.1}$$

where $m = \dim M$. Then, any sequence in $\mathcal{F}$ causes a **bubbling***: Namely, for any sequence $\{\varphi_i\} \in \mathcal{F}$, there exist a finite set $\mathcal{S}$ in M, say, $\mathcal{S} = \{x_1, \cdots, x_\ell\}$, and a smooth biharmonic map $\varphi_\infty : (M\backslash\mathcal{S}, g) \to (N, h)$ suth that,*

(1) a subsequence φ_{i_j} converges φ_∞ in the C^∞-topology on $M\backslash\mathcal{S}$, as $j \to \infty$, and

(2) the Radon measures $|d\varphi_{i_j}|^m\, v_g$ converges to a measure

$$|d\varphi_\infty|^m\, v_g + \sum_{i=1}^{\ell} a_k\, \delta_{x_k}, \tag{5.2}$$

as $j \to \infty$. Here a_k is a constant, and δ_{x_k} is the Dirac measure whose support is $\{x_k\}$ $(k = 1 \cdots, \ell)$.

As a corollary, we have immediately

Theorem 5.2. (*Bubbling for Harmonic Maps*) *Let (M, g) and (N, h) be two compact Riemannian manifolds. For every positive constant $C > 0$, let us consider a family of biharmonic maps of (M, g) into (N, h),*

$$\mathcal{F}^h = \left\{ \varphi : (M, g) \to (N, h),\ harmonic \mid \int_M |\, d\varphi \,|^m \, v_g \leq C \right\}, \tag{5.3}$$

where $m = \dim M$. Then, any sequence in $\mathcal{F}^h$ causes a **bubbling***: Namely, for any sequence $\{\varphi_i\} \in \mathcal{F}^h$, there exist a finite set $\mathcal{S}$ in M, say, $\mathcal{S} = \{x_1, \cdots, x_\ell\}$, and a smooth harmonic map $\varphi_\infty : (M \backslash \mathcal{S}, g) \to (N, h)$ suth that,*

(1) a subsequence φ_{i_j} converges φ_∞ in the C^∞-topology on $M \backslash \mathcal{S}$, as $j \to \infty$, and

(2) the Radon measures $|d\varphi_{i_j}|^m \, v_g$ converges to a measure

$$|d\varphi_\infty|^m \, v_g + \sum_{i=1}^{\ell} a_k \, \delta_{x_k}, \tag{5.4}$$

as $j \to \infty$. Here a_k is a constant, and δ_{x_k} is the Dirac measure whose support is $\{x_k\}$ $(k = 1 \cdots, \ell)$.

Proof. For any sequence in $\{\varphi_i\} \in \mathcal{F}^h$, the limit φ_∞ in Theorem 5.1 is a smooth biharmonic map of $(M \backslash \mathcal{S}, g)$ into (N, h). Due to (1) of Theorem 5.1, φ_{i_j} converges to φ_∞ in the C^∞-topology on $M \backslash \mathcal{S}$, so that $\tau(\varphi_{i_j})$ converges to $\tau(\varphi_\infty)$ pointwise on $M \backslash \mathcal{S}$. Since $\tau(\varphi_{i_j}) \equiv 0$, we have $\tau(\varphi_\infty) \equiv 0$ on $M \backslash \mathcal{S}$, i.e., φ_∞ is harmonic on $M \backslash \mathcal{S}$. And (1) and (2) hold also due to Theorem 5.1. □

6. Basic Inequalities

To prove Theorem 5.1, it is necessary to prepare the following two basic inequalities.

Lemma 6.1. *Assume that the sectional curvature of (N, h) is bounded above by a constant C. Then, we have*

$$\frac{1}{2} \Delta \, |V|^2 + C \, |d\varphi \,| \, |V|^2 \geq |\overline{\nabla} V|^2 \tag{6.1}$$

for all $V \in \Gamma(\varphi^{-1} TN)$.

Proof. Indeed, let us recall (3.2)

$$\frac{1}{2}\,\Delta\,|V|^2 = h(-\mathcal{R}(V), V) + |\,\overline{\nabla} V\,|^2, \tag{6.2}$$

for all $V \in \Gamma(\varphi^{-1}TN)$. Since

$$h(\mathcal{R}(V), V) = \sum_{i=1}^{m} h(R^N(V, d\varphi(e_i))d\varphi(e_i), V),$$

the right hand side of (6.2) is bigger than or equal to

$$-C\sum_{i=1}^{m} |d\varphi(e_i)|^2\,|V|^2 + |\,\overline{\nabla} V\,|^2 = -C\,|d\varphi|^2\,|V|^2 + |\,\overline{\nabla} V\,|^2.$$

We have (6.1). □

Lemma 6.2. *Under the same assumption as Lemma 6.1, we have*

$$|\tau(\varphi)|\,\Delta\,|\tau(\varphi)| + C\,|d\varphi\,|^2\,|\tau(\varphi)|^2 \geq 0 \tag{6.3}$$

for all $\varphi \in C^\infty(M, N)$*.*

Proof. The proof goes in the same way as Lemma 3.2. Indeed, by the equality of (3.5) in the proof of Lemma 6.1 and also Lemma 6.1 itself, we have

$$\begin{aligned}
|\tau(\varphi)|\,\Delta\,|\tau(\varphi)| &+ |\,\nabla\,|\tau(\varphi)|\,|^2 + C\,|d\varphi\,|^2\,|\tau(\varphi)|^2 \\
&\geq \frac{1}{2}\Delta\,|\tau(\varphi)| + C\,|d\varphi\,|^2\,|\tau(\varphi)|^2 \\
&\geq |\overline{\nabla} V|^2.
\end{aligned} \tag{6.4}$$

So that we have

$$\begin{aligned}
|\tau(\varphi)|\,\Delta\,|\tau(\varphi)| &+ C\,|d\varphi\,|^2\,|\tau(\varphi)|^2 \\
&\geq |\overline{\nabla} V|^2 - |\,\nabla\,|\tau(\varphi)|\,|^2 \\
&\geq 0
\end{aligned} \tag{6.5}$$

due to (3.6) in the proof of Lemma 3.2. □

Proposition 6.3. *Assume that the sectional curvature of* (N, h) *is bounded above by a positive constant* $C > 0$*. Then, there exists a positive number* $\epsilon_0 > 0$ *depending only on the Sobolev constant of* (M, g) *and* C *such that for every* $\varphi \in C^\infty(M, N)$*, if*

$$\int_{B_r(x_0)} |d\varphi|^m\, v_g \leq \epsilon_0, \tag{6.6}$$

then

$$\sup_{B_{r/2}(x_0)} |\tau(\varphi)|^2 \leq \frac{C'}{r^{m/2}} \int_{B_r(x_0)} |\tau(\varphi)|^2 \, v_g. \tag{6.7}$$

for some positive constant $C' > 0$ *depending only on* C *and* $m = \dim M$.

Proof. The proof of Proposition 6.3 goes in the same line of the one of Theorem 3.3. We retain the situation in Section Three. Multiply $|\tau(\varphi)|^{p-2}\,\eta^2$ to both hand sides of (6.3) and integrate it over M. Then, we have

$$0 \leq \int_M |\tau(\varphi)|^{p-1}\,\eta^2\,\Delta(|\tau(\varphi)|)\,v_g + C\int_M |d\varphi|^2\,|\tau(\varphi)|^p\,\eta^2\,v_g. \tag{6.8}$$

In order to estimate the second term of the right hand side of (6.8), we need the following lemma.

Lemma 6.4. *We have*

$$\begin{aligned} &\int_M |d\varphi|^2\,|\tau(\varphi)|^p\,\eta^2\,v_g \\ &\qquad \leq C'' \left\{ \int_{B_r(x_0)} |d\varphi|^m\,v_g \right\}^{2/m} \int_M |\nabla(\,|\tau(\varphi)|^{p/2}\,\eta)\,|^2\,v_g, \end{aligned} \tag{6.9}$$

where $C'' > 0$ *is a positive constant independent on* φ.

We postpone giving a proof of Lemma 6.4, and we continue the proof of Proposition 6.3.

In the first step of the proof of Theorem 3.3, we have instead of (4.2), by (6.8),

$$\begin{aligned} 0 &\leq \int_M |\tau(\varphi)|^{p-1}\,\eta^2\,\Delta(|\tau(\varphi)|)\,v_g + C\int_M |d\varphi|^2\,|\tau(\varphi)|^p\,\eta^2\,v_g \\ &= -\frac{4(p-1)}{p^2}\int_M |\nabla(|\tau(\varphi)|^{p/2})|^2\,\eta^2\,v_g \\ &\quad -\frac{4}{p}\int_M g(\eta\,\nabla(|\tau(\varphi)|^{p/2}), |\tau(\varphi)|^{p/2}\,\nabla\eta)\,v_g \\ &\quad + C\int_M |d\varphi|^2\,|\tau(\varphi)|^p\,\eta^2\,v_g. \end{aligned} \tag{6.10}$$

By the same argument as in (4.3), (4.4) and (4.5), (4.5) is changed into the following:

$$\int_M \eta^2 \, | \nabla(|\tau(\varphi)|^{p/2})|^2 \, v_g \leq \frac{p^2}{(p-1)^2} \int_M |\tau(\varphi)|^p \, |\nabla \eta|^2 \, v_g + C \int_M |d\varphi|^2 \, |\tau(\varphi)|^p \, \eta^2 \, v_g. \tag{6.11}$$

And then, by the same as in (4.6), we have,

$$\begin{aligned}
&\int_M | \nabla(|\tau(\varphi)|^{p/2} \, \eta) \, |^2 \, v_g \\
&\leq 4 \, \frac{p^2}{(p-1)^2} \int_M |\tau(\varphi)|^p \, |\nabla \eta|^2 \, v_g + C \int_M |d\varphi|^2 \, |\tau(\varphi)|^p \, \eta^2 \, v_g \\
&\leq 4 \, \frac{p^2}{(p-1)^2} \int_M |\tau(\varphi)|^p \, |\nabla \eta|^2 \, v_g \\
&\quad + CC'' \left\{ \int_{B_r(x_0)} |d\varphi|^m \, v_g \right\}^{2/m} \int_M |\nabla(\, |\tau(\varphi)|^{p/2} \, \eta) \, |^2 \, v_g,
\end{aligned} \tag{6.12}$$

instead of (4.6). In the last inequality, we used (6.9) in Lemma 6.4.

Assume that

$$\int_{B_r(x_0)} |d\varphi|^m \, v_g \leq \epsilon_0. \tag{6.13}$$

Then, due to (6.12), we have

$$\int_M | \nabla(|\tau(\varphi)|^{p/2} \, \eta) \, |^2 \, v_g \leq 4 \, \frac{p^2}{(p-1)^2} \int_M |\tau(\varphi)|^p \, |\nabla \eta|^2 \, v_g + CC'' {\epsilon_0}^{2/m} \int_M |\nabla(\, |\tau(\varphi)|^{p/2} \, \eta) \, |^2 \, v_g. \tag{6.14}$$

If we take $\epsilon_0 > 0$ enough small such as $1 - CC'' \, {\epsilon_0}^{2/m} > \frac{1}{2}$, i.e., $\frac{1}{(2CC'')^{2/m}} > \epsilon_0$, then, we have

$$\int_M | \nabla(|\tau(\varphi)|^{p/2} \, \eta) \, |^2 \, v_g \leq 8 \, \frac{p^2}{(p-1)^2} \int_M |\tau(\varphi)|^p \, |\nabla \eta|^2 \, v_g. \tag{6.15}$$

Now, the proof of Theorem 3.3 works in the same way, and then we obtain Proposition 6.3. □

(*Proof of Lemma 6.4*) We may assume that the support of the cutoff function η is contained in $B_r(x_0)$, and then we use the Hölder

inequality of this type on $B_r(x_0)$:

$$\int_{B_r(x_0)} F\, G\, g_g \leq \left(\int_{B_r(x_0)} F^{m/2}\, v_g\right)^{2/m} \left(\int_{B_r(x_0)} G^{m/(m-2)}\, v_g\right)^{(m-2)/m}.$$

Then, we have

$$\begin{aligned}\int_M |d\varphi|^2\, |\tau(\varphi)|^p\, \eta^2\, v_g &= \int_{B_r(x_0)} |d\varphi|^2\, (\,|\tau(\varphi)|^{p/2}\, \eta\,)^2\, v_g \\ &\leq \left(\int_{B_r(x_0)} (\,|d\varphi|^2\,)^{m/2}\, v_g\right)^{2/m} \times \\ &\quad \times \left(\int_{B_r(x_0)} (\,(\,|\tau(\varphi)|^{p/2}\, \eta\,)^2\,)^{m/(m-2)}\, v_g\right)^{(m-2)/m} \\ &= \left(\int_{B_r(x_0)} (\,|d\varphi|^2\,)^{m/2}\, v_g\right)^{2/m} \times \\ &\quad \times \left(\int_M (\,|\tau(\varphi)|^{p/2}\, \eta\,)^{2m/(m-2)}\, v_g\right)^{(m-2)/m} \\ &\leq C_0 \left(\int_{B_r(x_0)} |d\varphi|^m\, v_g\right)^{2/m} \int_M |\nabla(\,|\tau(\varphi)|^{p/2}\, \eta\,)|^2\, v_g,\end{aligned} \tag{6.16}$$

where in the last inequality of (6.16), we used the Sobolev inequality for $F = |\tau(\varphi)|^{p/2}\, \eta$:

$$\left(\int_M F^{2m/(m-2)}\, v_g\right)^{(m-2)/m} \leq C_0 \int_M |\nabla F|^2.$$

We obtain (6.9). □

7. Proof of Theorem 5.1

Now we are in position giving a proof of Theorem 5.1.

Take any sequence $\{\varphi_i\}$ in $\mathcal{F}$. For the $\epsilon_0 > 0$ in Propisition 6.3, and let us consider

$$\mathcal{S} := \left\{ x \in M |\ \liminf_{i\to\infty} \int_{B_r(x)} |d\varphi_i|^m\, v_g \geq \epsilon_0 \quad (\text{for all }\ r > 0) \right\}. \tag{7.1}$$

Then, the set $\mathcal{S}$ is finite. Because, for every finite subset $\{x_i\}_{i=1}^k$ in $\mathcal{S}$, let us take a sufficiently small positive number $r_0 > 0$ in such a way that $B_{r_0}(x_i) \cap B_{r_0}(x_j) = \emptyset \quad (i \neq j)$. Then, we have for a sufficiently

large i,

$$\begin{aligned} k\,\epsilon_0 &\leq \sum_{j=1}^{k} \int_{B_{r_0}(x_j)} |d\varphi_i|^m\, v_g \\ &= \int_{\cup_{j=1}^{k} B_{r_0}(x_j)} |d\varphi_i|^m\, v_g \\ &\leq \int_M |d\varphi_i|^m\, v_g \\ &\leq C < \infty \end{aligned} \tag{7.2}$$

by definition of $\mathcal{F}$. Thus, we have

$$k \leq \frac{C}{\epsilon_0},$$

which implies that $\#\mathcal{S} \leq \frac{C}{\epsilon_0} < \infty$.

Then, if necessary by taking a subsequence of $\{\varphi_i\}$, we may assume that

$$\mathcal{S} = \left\{ x \in M |\ \limsup_{i\to\infty} \int_{B_r(x)} |d\varphi_i|^m\, v_g \geq \epsilon_0 \right\}. \tag{7.3}$$

Because, if not so, let us denote the right hand side of (7.3) by $\overline{\mathcal{S}}$. Then, by definition, $\mathcal{S}$ is a proper subset of $\overline{\mathcal{S}}$. Take a point $\overline{x} \in \overline{\mathcal{S}}\backslash\mathcal{S}$. By taking a subsequence of $\{\varphi_i\}$, by the same letter, in such a way that

$$\liminf_{i\to\infty} \int_{B_r(\overline{x})} |d\varphi_i|^m\, v_g \geq \epsilon_0,$$

For this $\{\varphi_i\}$, $\overline{x}$ belongs to $\mathcal{S}$. Since $\mathcal{S}$ is a finite set, this process stops at finite times, then at last we have $\overline{\mathcal{S}} = \mathcal{S}$.

Now, let $x \in M\backslash\mathcal{S}$. Then,

$$\limsup_{i\to\infty} \int_{B_r(x)} |d\varphi_i|^m\, v_g < \epsilon_0. \tag{7.4}$$

Due to Proposition 6.3 and the definition of $\mathcal{F}$, we have

$$\begin{aligned} \sup_{B_{r/2}(x)} |\tau(\varphi_i)|^2 &\leq \frac{C}{r^{m/2}} \int_{B_r(x)} |\tau(\varphi_i)|^2\, v_g \\ &\leq \frac{C^2}{r^{m/2}}, \end{aligned} \tag{7.5}$$

so that we have that

(C^0): the C^0-estimate on $B_r(x)$ of $\tau(\varphi_i)$ uniformly on i.

On the other hand, since $\varphi_i \in \mathcal{F}$, all φ_i are biharmonic, i.e., φ_i satisfy that the equations

$$\tau_2(\varphi_i) = \overline{\Delta}(\tau(\varphi_i)) - \mathcal{R}(\tau(\varphi_i)) = 0 \tag{7.6}$$

$$\Longleftrightarrow \begin{cases} (1) \quad \overline{\Delta}\sigma_i = \mathcal{R}(\sigma_i), \\ (2) \quad \tau(\varphi_i) = \sigma_i. \end{cases} \tag{7.7}$$

Notice that both the (1) and (2) of (7.7) are the (non-linear) elliptic partial differential equations. Due to (1), the C^0-estimate for σ_i means that the C^∞-estimate of σ_i, and due to (2), the C^∞-estimate of σ_i means that the C^∞-estimate of φ_i. Thus, due to the estimate (C^0) above, we obtain the C^∞-estimates on $B_r(x)$ of φ_i uniformly on i. Therefore, there exists a subsequence $\{\varphi_{i_j}\}$ of $\{\varphi_i\}$ and a smooth map $\varphi_\infty : M\backslash\mathcal{S} \to N$ such that φ_{i_j} converges to φ_∞ on $B_r(x)$ in the C^∞-topology as $j \to \infty$. Thus, $\varphi_\infty : (M\backslash\mathcal{S}, g) \to (N, h)$ is also biharmonic.

For (2) in Theorem 5.1, let us consider the Radon measures $|d\varphi_{i_j}|^m\, v_g$. Then, these have a weak limit which is also a Randon measure, say μ. Recall that μ is by definition a *Radon measure* if (1) μ is locally finite, i.e., $\mu(K) < \infty$ for every compact subset K on M, and (2) μ is Borel regular, i.e., it holds that, for all Borel subset A of M,

$$\mu(A) = \sup\{\mu(K) |\text{ for all compact subset } K \text{ of } A\}, \text{ and}$$
$$\mu(A) = \inf\{\mu(O) |\text{ for all open subset } O \text{ of } M \text{ including } A\}.$$

On the other hand, since φ_{i_j} converges to φ_∞ on $M\backslash\mathcal{S}$ in the C^∞-topology as $j \to \infty$, it holds that

$$\mu = |d\varphi_\infty|^m\, v_g \quad \text{on } M\backslash\mathcal{S}. \tag{7.8}$$

Here, $\mathcal{S}$ is a finite subset of M, say $\mathcal{S} = \{x_1, \cdots, x_k\}$. Then, the Radon measure $\mu - |d\varphi_\infty|^m\, v_g$ satisfies that its support contains in $\mathcal{S}$. Therefore, it holds that

$$\mu - |d\varphi_\infty|^m\, v_g = \sum_{i=1}^k a_j\, \delta_{x_j} \tag{7.9}$$

for some non-negative real numbers a_j $(j = 1, \cdots, k)$ and δ_{x_j} is the Dirac measure which satisfies by definition

$$\delta_{x_j}(A) = \begin{cases} 1 & (x_j \in A), \\ 0 & (x_j \notin A), \end{cases}$$

for every Borel subset A of M. Remark here that $a_j < \infty$ for every $j = 1, \cdots, k$. Because μ is a Radon measure, so that μ is locally finite.

Therefore, the Radon measure $|d\varphi_{i_j}|^m\, v_g$ converges weakly to μ, and

$$\mu = |d\varphi_\infty|\, v_g + \sum_{j=1}^{k} a_j\, \delta_{x_j} \tag{7.10}$$

due to (7.9). We have (2) of Theorem 5.1. □

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